1/

Math 112: Introductory Real Analysis

§ Lecture 3 (Feb 3, 2025)

$600 for the semester

Advertise a compensated notetaker position. Interested students should contact daopeernotetaking@fas.harvard.edu

Last time: fields, ordered fields, the real field $\mathbb{R}$, Archimedean property of $\mathbb{R}$, $\mathbb{Q}$ is dense in $\mathbb{R}$.

Today: the complex field & Euclidean spaces

more relevant to complex analysis

more relevant to multi-variable calculus

- **Def** A complex number is an ordered pair of real numbers

  (Idea: Think of (a,b) as $a+bi$, with $i=\sqrt{-1}$)

  Denote the set of all complex numbers by $\mathbb{C}$.

  Define $(a,b)+(c,d) := (a+c,\ b+d)$

  and $(a,b)\cdot(c,d) := (ac-bd,\ ad+bc)$.

Thm These definitions of addition and multiplication turn the set $\mathbb{C}$ into a field, with $0:=(0,0)$ and $1:=(1,0)$.

proof) exercise ■

Note, $\mathbb{C}$ contains $\mathbb{R}$ as a subfield, as

$$\begin{aligned} \mathbb{R} &\hookrightarrow \mathbb{C} \\ a &\longmapsto (a,0) \end{aligned}$$

<u>Def</u> $i := (0,1)$.

Then, $i^2 = (0,1)\cdot(0,1) = (-1,0) = -1$,

and $(a,b) = (a,0) + (b,0)(0,1) = a + bi$.

While $\mathbb{R}$ is an ordered field, $\mathbb{C}$ <u>cannot</u> be equipped with an order.

~~extending~~

(This is because $i^2 = -1 < 0$.)

The benefit of using $\mathbb{C}$ instead of $\mathbb{R}$ is that it is <u>algebraically closed</u>.

<u>Thm</u> (Fundamental theorem of algebra; $\mathbb{C}$ is algebraically closed)

Every non-constant polynomial $p(z) \in \mathbb{C}[z]$ has a root in $\mathbb{C}$.

(The typical proof uses continuity of $|p(z)|$, a notion we haven't introduced yet.
We'll come back to this later in this course, if time permits.
A proof can be found in Chapter 8 of Rudin.)

- Euclidean spaces

Def For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples

$$\mathbf{x} = (x_1, \cdots, x_k)$$

of real numbers.

The elements of $\mathbb{R}^k$ are called points (in the Euclidean space $\mathbb{R}^k$), or vectors.

Def Let $\mathbb{k}$ be a field.

A vector space over $\mathbb{k}$ (or a $\mathbb{k}$-vector space) is a set V equipped with two operations, addition and scalar multiplication, satisfying the following vector space axioms:

(Axioms for addition)

(commutativity) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ for all $\mathbf{x},\mathbf{y} \in V$

(associativity) $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ for all $\mathbf{x},\mathbf{y},\mathbf{z} \in V$

(unit) There is $\mathbf{0} \in V$ such that $\mathbf{0}+\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$

(inverse) For every $\mathbf{x} \in V$, there is $-\mathbf{x} \in V$ such that $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$.

(Axioms for scalar multiplication)

(associativity) $a(b\mathbf{x}) = (ab)\mathbf{x}$ for all $a,b \in \mathbb{k}$ and $\mathbf{x} \in V$

(unit) $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$

(Distributivity axioms)

(1) $a(\mathbf{x}+\mathbf{y}) = a\mathbf{x}+a\mathbf{y}$ for all $a \in \mathbb{k}$, $\mathbf{x},\mathbf{y} \in V$

(2) $(a+b)\mathbf{x} = a\mathbf{x}+b\mathbf{x}$ for all $a,b \in \mathbb{k}$, $\mathbf{x} \in V$.

4/

E.g. $\mathbb{R}^k$ is a vector space over $\mathbb{R}$.

Def Define the inner product of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ by

$$\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^{k} x_i y_i ,$$

and the norm of $\mathbf{x} \in \mathbb{R}^k$ by

$$|\mathbf{x}| := (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}} = \left(\sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}} .$$

Thm If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$, then

(a) $|\mathbf{x}| \geq 0$

(b) $|\mathbf{x}| = 0$ iff $\mathbf{x} = 0$

(c) $|\alpha \mathbf{x}| = |\alpha| \, |\mathbf{x}|$

(d) $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| \, |\mathbf{y}|$

(e) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$

(f) $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$ ← triangle inequality

proof) Exercise.

We can think of $|\mathbf{x} - \mathbf{y}|$ as the "distance" between $\mathbf{x}$ and $\mathbf{y}$.
A set where we can talk about "distances" among its elements is called a metric space, which we'll discuss probably next week.
($\mathbb{R}^k$ is an example of a metric space)